**Mavzu: Aniq integral va uning tatbiqlari.**

**Reja:**

1. Aniq integral, uning geometrik ma‘nosi.
2. Aniq integralning asosiy xossalari.
3. Nyuton-Leybnits formulasi.
4. Aniq integralning tatbiqlari.

***Tayanch tushunchalar***: aniq integral, xossalari, hisoblash usullari, quyi va yuqori chegaralar, Nyuton-Leybnits formulasi, tatbiqlari

**Aniq integral, uning geometrik ma’nosi.** Aniq integral - matematik analizning eng muhim tushunchalaridan biridir. Egri chiziq bilan chegaralangan yuzalarni, egri chiziqli yoylar uzunliklarini, hajmlarni, bajarilgan ishlarni, yo’llarni, inersiya momentlarini va hokazolarni hisoblash masalasi shu tushuncha bilan bog’liq.

$[a, b]$ kesmada $y = f(x)$ uzluksiz funksiya berilgan bo’lsin. Quyidagi amallarni bajaramiz:

1. $[a, b]$ kesmani qo’yidagi nuqtalar bilan ixtiyoriy n ta qismga bo’lamiz, va ularni qismiy intervallar deb ataymiz:

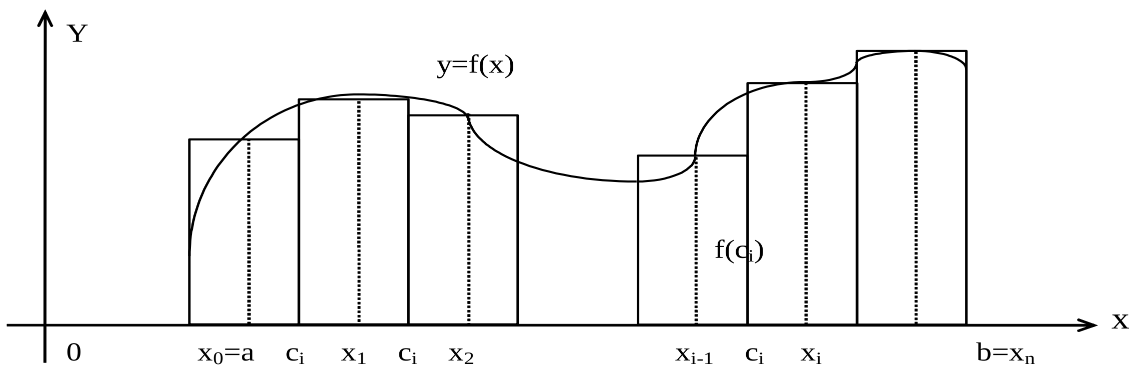
$$a = x_0 < x_1 < x_2 < x_3 <\dots x_{i-1} < x_i \dots < x_n = b$$

2. Qismiy intervallarning uzunliklarini bunday belgilaymiz: [1]

$$\Delta x_1 = x_1 - a \quad \Delta x_2 = x_2 - x_1 \dots \quad \Delta x_i = x_i - x_{i-1} \dots \quad \Delta x_n = b - x_{n-1}$$

$\sigma_n$ yig’indi $f(x)$ funksiya uchun $[a, b]$ kesmada tuzilgan integral yig’indi deb ataladi. $\sigma_n$ integral yig’indi qisqacha bunday yoziladi:

$$\sigma_n = \sum_{i=1}^{n} f(c_i)\Delta x_i \, .$$




[1] *Larson Edvards. /Calculus/ 2010. 272,226 betlarning mazmun, mohiyatidan foydalanildi.*

1-rasm.

Integral yig'indining geometric ma'nosi ravshan: Agar $f(x) \geq 0$ bo'lsa, u holda $\sigma_n$ – asoslari $\Delta x_1$, $\Delta x_2$, ..., $\Delta x_i$,...,$\Delta x_n$ va balandliklari mos ravishda $f(c_1)$, $f(c_2)$, ..., $f(c_i)$, ..., $f(c_n)$ bo'lgan to'g'ri to'rtburchak yuzlarining yig'indisidan iborat (1-rasm).

Endi bo'lishlar soni n ni orttira boramiz ($n\to\infty$) va bunda eng katta intervalning uzunligi nolga intilishini, ya'ni max $\Delta x_i\to 0$ deb faraz qilamiz.

Ushbu ta'rifni beramiz.

**Ta'rif.** Agar $\sigma_n$ integral yig'indi $[a, b]$ kesmani qismiy $[x_i, x_{i-1}]$ kesmalarga ajratish usuliga va ularning har biridan $c_i$ nuqtani tanlash usuliga bog'liq bo'lmaydigan chekli songa intilsa, u holda shu son $[a, b]$ kesmada $f(x)$ funksiyadan olingan aniq integral deyiladi va bunday belgilanadi:

$$\int_a^b f(x)dx.$$

Bu yerda $f(x)$ integral ostidagi funksiya. $[a, b]$ kesma integrallash oralig'i, $a$ va $b$ sonlar integrallashning qo'yi va yuqori chegarasi deyiladi.

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$ [2]

Aniq integralning ta'rifidan ko'rinadiki, aniq integral hamma vaqt mavjud bo'lavermas ekan. Biz qo'yida aniq integralning mavjudlik teoremasini isbotsiz keltiramiz.[3]

**Teorema.** Agar $u=f(x)$ funksiya $[a, b]$ kesmada uzluksiz bo'lsa, u integrallanuvchidir, ya'ni bunday funksiyaning aniq integrali mavjuddir.[4]

Agar yuqoridan $y=f(x)\geq 0$ funksiyaning grafigi, qo'yidan OX o'qi, yon tomonlaridan esa $x=a$, $x=b$ to'g'ri chiziqlar bilan chegaralangan sohani egri chiziqli trapetsiya deb atasak, u holda

$$\int_a^b f(x)dx$$

Aniq integralning geometrik ma'nosi ravshan bo'lib qoladi: $f(x)\geq 0$ bo'lganda u shu egri chiziqli trapetsiyaning yuziga son jihatdan teng bo'ladi.

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[2]*Larson Edvards. /Calculus/ 2010. 273,226 betlarning mazmun, mohiyatidan foydalanildi.*

[3]*Larson Edvards. /Calculus/ 2010. 273, 226- betlarning mazmun, mohiyatidan foydalanildi.*

[4]*Larson Edvards. /Calculus/ 2010. 272, 226- betlarning mazmun, mohiyatidan foydalanildi.*

**1-izoh.** Aniq integralning qiymati funksiyaning ko'rinishiga va integrallash chegarasiga bog'liq.

Masalan: $\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(z)dz.$

**2-izoh.** Aniq integralning chegaralari almashtirilsa, integralning ishorasi o'zgaradi.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

**3-izoh.** Agar aniq integralning chegaralari teng bo'lsa, har qanday funksiya uchun ushbu tenglik o'rinli bo'ladi:

$$\int_a^b f(x)dx = 0$$

**Aniq integralning asosiy xossalari.**

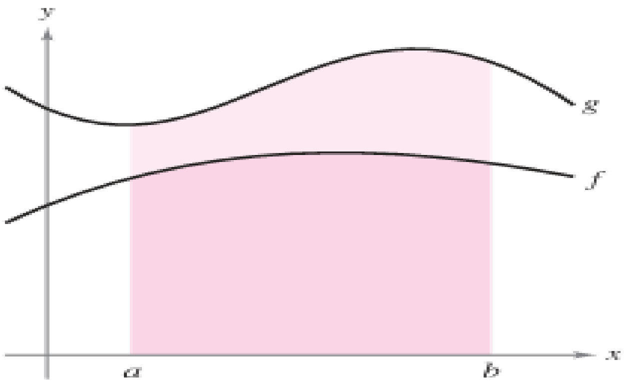
**1-xossa**. O'zgarmas ko'paytuvchini aniq integral belgisidan tashqariga chiqarish mumkin: $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

**2-xossa.** Bir nechta funksiyaning algebraik yig'indisining aniq integrali qo'shiluvchilar integralining yig'indisiga teng (ikki qo'shiluvchi bo'lgan hol bilan chegaralanamiz):

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

**3-xossa.** Agar [*a*, *b*] kesmada ikki *f*(*x*) va g(*x*) funksiya (*a*<*b*) $f(x) \leq g(x)$ shartni qanoatlantirsa, ushbu tengsizlik o'rinli:

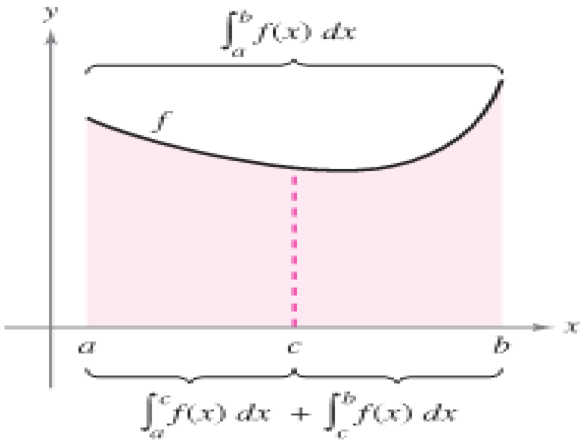
$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$




$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

**4-xossa.** Agar [*a*, *b*] kesma bir necha qismga bo'linsa, u holda [*a*, *b*] kesma bo'yicha aniq integral har bir qism bo'yicha olingan aniq integrallar yig'indisiga teng. [*a*, *b*] kesma ikki qismga bo'lingan hol bilan cheklanamiz, ya'ni $a<c<b$ bo'lsa, u holda

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$




**5-xossa**. Agar *m* va *M* sonlar *f*(*x*) funksiyaning [*a*, *b*] kesmada eng kichik va eng katta qiymati bo'lsa, ushbu tengsizlik o'rinli.

$$m(b-a) \le \int_a^b f(x)dx. \le M(b-a) \qquad (5)$$

**Isboti**. Shartga ko'ra $m \le f(x) \le M$ ekani kelib chiqadi. 3-xossaga asosan qo'yidagiga ega bo'lamiz:

$$\int_a^b mdx \le \int_a^b f(x)dx \le \int_a^b Mdx \qquad (5^*)$$

Biroq

$$\int_a^b m dx = m\int_a^b dx = m \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \Delta x_i = m(b-a)$$

$$\int_a^b M dx = M\int_a^b dx = M \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \Delta x_i = M(b-a)$$

bo'lgani uchun (5*) tengsizlik

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

bo'ladi.

**6-xossa** (o'rta qiymat haqidagi teorema).

Agar $f(x)$ funksiya $[a, b]$ kesmada uzluksiz bo'lsa, bu kesmaning ichida shunday $x=s$ nuqta topiladiki, bu nuqtada funksiyaning qiymati uning shu kesmadagi o'rtacha qiymatiga teng bo'ladi, ya'ni

$$f(s) = \frac{1}{(b-a)}\int_a^b f(x)dx.$$

**Isboti.** Faraz qilaylik, $m$ va $M$ sonlar $f(x)$ uzluksiz funksiyaning $[a, b]$ kesmadagi eng kichik va eng katta qiymati bo'lsin. Aniq integralni baholash haqidagi xossaga ko'ra qo'yidagi qo'sh tengsizlik to'g'ri:

$$m(b-a) \le \int_a^b f(x)dx. \le M(b-a)$$

tengsizlikning hamma qismlarini $b$-$a$>0 ga bo'lamiz. Natijada

$$m \le \frac{1}{(b-a)}\int_a^b f(x)dx \le M$$

ni hosil qilamiz. Ushbu $\mu = \frac{1}{(b-a)}\int_a^b f(x)dx.$ belgilashni kiritib, qo'sh tengsizlikni qayta yozamiz. $m \le \mu \le M$

$f(x)$ funksiya $[a, b]$ kesmada uzluksiz bo'lgani uchun u $m$ va $M$ orasidagi hamma oraliq qiymatlarni qabul qiladi.

Demak, biror $x=s$ qiymatda $\mu=f(s)$ bo'ladi, ya'ni

$$f(s) = \frac{1}{(b-a)}\int_a^b f(x)dx.$$ [5]

1. [5]*Larson Edvards. /Calculus/ 2010. 276, 226 betlarning mazmun, mohiyatidan foydalanildi.*

Teorema isbotlandi.

**Nyuton-Leybnits formulasi.** Aniq integrallarni integral yig'indining limiti sifatida bevosita hisoblash ko'p hollarda juda qiyin, uzoq hisoblashlarni talab qiladi va amalda juda kam qo'llaniladi. Integrallarni topish formulasi Nyuton-Leybnits teoremasi bilan beriladi.

Teorema. Agar *F(x)* funksiya *f(x)* funksiyaning $[a, b]$ kesmadagi boshlang'ich funksiyasi bo'lsa, u holda aniq integral boshlang'ich funksiyaning integrallash oralig'idagi orttirmasiga teng, ya'ni

$$\int_a^b f(x)dx = F(b) - F(a) \qquad (1)$$

(1) tenglik Nyuton-Leybnits formulasi deyiladi.[6]

**Isboti.** *F(x)* funksiya *f(x)* funksiyaning biror boshlang'ich funksiyasi bo'lsin, u holda 1-teoremaga ko'ra $\int_a^x f(t)dt$ funksiya ham *f(x)* funksiyaning boshlang'ich funksiyasi bo'ladi. Berilgan funksiyaning ikkita istalgan boshlang'ich funksiyalari bir-biridan o'zgarmas *C* qo'shiluvchiga farq qiladi, ya'ni F*(x)=F(x)+C.*

Shuning uchun:

$$\int_a^x f(t)dt = F(x) + C$$

*C*-o'zgarmas miqdorni aniqlash uchun bu tenglikda x=a deb olamiz:

$$\int_a^a f(t)dt = F(a) + C \text{ , } \int_a^a f(t)dt = 0$$

bo'lgani uchun *F(a)+C*=0. Bundan, *S*=-*F(a)*. Demak, $\int_a^x f(t)dt = F(x) - F(a)$

Endi *x*=*b* deb Nyuton-Leybnits formulasini hosil qilamiz:

$$\int_a^b f(t)dt = F(b)\text{-}F(a)$$

yoki integrallash o'zgaruvchisini *x* bilan almashtirsak:

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2. [6]*Larson Edvards. /Calculus/ 2010. 283betning mazmun mohiyatidan foydalanildi.*

$$\int_a^b f(x)dx = F(b)\text{-}F(a)$$

$F(b)\text{-}F(a) = F(x)|_a^b$ belgilash kiritib, oxirgi formulani qo'yidagicha qayta yozish mumkin:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b)\text{-}F(a)$$

Teorema isbotlandi.

Integral ostidagi funksiyaning boshlang'ich funksiyasi ma'lum bo'lsa, u holda Nyuton-Leybnits formulasi aniq integrallarni hisoblash uchun amalda qulay usulni beradi. Faqat shu formulaning kashf etilishi aniq integralni hozirgi zamonda matematik analizda tutgan o'rnini olishga imkon bergan. Nyuton-Leybnits formulasi aniq integralning tatbiqi sohasini ancha kengaytirdi, chunki matematika bu formula yordamida xususiy ko'rinishdagi turli masalalarni yechish uchun umumiy usulga ega bo'ldi.

***Misollar.***

1) $\displaystyle\int_0^1 \frac{dx}{1+x^2} = arctgx|_0^1 = arctg1 - arctg0 = \frac{\pi}{4}$

2) $\displaystyle\int_{\sqrt{3}}^{\sqrt{8}} \frac{xdx}{\sqrt{1+x^2}} = \frac{1}{2}\int_{\sqrt{3}}^{\sqrt{8}} \frac{d(1+x^2)}{\sqrt{1+x^2}} = \frac{1}{2}\int_{\sqrt{3}}^{\sqrt{8}} (1+x^{2)^{-\frac{1}{2}}} d(1+x^2) = (1+x^2)^{\frac{1}{2}}|_{\sqrt{3}}^{\sqrt{8}} =$

$= \sqrt{9} - \sqrt{4} = 3 - 2 = 1.$

3) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 xdx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x)dx = \frac{1}{2}(x - \frac{1}{2}\sin 2x)|_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$

**O'zgaruvchini almashtirish.**

Bizga $\displaystyle\int_a^b f(x)dx$ aniq integral berilgan bo'lsin, bunda $f(x)$ funksiya $[a, b]$ kesmada uzluksizdir.

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Aniq integral (2) formula bo'yicha hisoblaganda yangi o'zgaruvchidan eski o'zgaruvchiga qaytish kerak emas, balki eski o'zgaruvchining chegaralarini keyingi boshlang'ich funksiyaga qo'yish kerak. [7]

***Misollar.***

1) $\int_3^8 \frac{xdx}{\sqrt{x+1}}$ integralni hisoblang.

***Yechish.*** $x$+1=u$^2$ deb almashtirsak, $x$=u$^2$-1, $dx$=2u$du$ bo'ladi. Integrallashning yangi chegaralari: $x$=3 bo'lganda $t$=2.

$x$=8 bo'lganda u=3 u holda:

$$\int_3^8 \frac{xdx}{\sqrt{x+1}} = \int_2^3 \frac{(t^2-1)2udu}{t} = 2\int_2^3 (u^2-1)du = 2(\frac{u^3}{3}-u)\Big|_2^3 = 2(6-\frac{2}{3}) = \frac{32}{3};$$

2) $\int_0^1 \sqrt{1-x^2}dx$ integralni hisoblang.

***Yechish.*** $x=sinu$ deb almashtirsak, $dx$=cos$udu$, 1-$x^2$=cos$^2u$ bo'ladi. Integrallashning yangi chegaralarini aniqlaymiz: $x$=0 bo'lganda u=0

$x$=1 bo'lganda u=$\pi$/2 . U holda:

$$\int_0^1 \sqrt{1-x^2}dx = \int_0^{\pi/2} \cos^2 udu = \frac{1}{2}\int_0^{\pi/2} (1+\cos 2udu = \frac{1}{2}(u+\frac{1}{2}\sin 2u)\Big|_0^{\pi/2} = \frac{\pi}{4}$$

**Aniq integralni bo'laklab integrallash.**

Faraz qilaylik, $u(x)$ va $v(x)$ funksiyalar $[a, b]$ kesmada differensiallanuvchi funksiyalar bo'lsin. U holda: $(uv)' = u'v + uv'$

Bu tenglikni ikkala tomonini a dan b gacha bo'lgan oraliqda integrallaymiz.

$$\int_a^b (uv)'dx = \int_a^b u'vdx + \int_a^b uv'dx \qquad (3)$$

Lekin $\int (uv)'dx = uv + C$ bo'lgani sababli, $\int (uv)'dx = uv\Big|_a^b$

Demak, (3) tenglikni qo'yidagi ko'rinishda yozish mumkin:

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[7] ***Larson Edvards. /Calculus/ 2010. 303. 226- betning mazmun mohiyatidan foydalanildi.***

$$uv\Big|_a^b = \int_a^b vdu + \int_a^b udv$$

Bundan
$$\int_a^b udv = uv\,|_a^b - \int_a^b vdu \qquad (4)$$

Bu formula aniq integralni bo'laklab integrallash formulasi deyiladi.

***Misollar.***

1) $\int_0^1 arctgxdx$ integral hisoblansin.

$$\int_0^1 arctgxdx = \left| \begin{matrix} u = arctgx & du = \dfrac{dx}{1+x^2} \\ dv = dx & v = x \end{matrix} \right| = xarctgx|_0^1 - \int_0^1 \frac{xdx}{1+x^2} = arctg1 -$$

$$\frac{1}{2}\ln\left(1+x^2\right)\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2 .$$

2) $\int_0^1 xe^{-x}dx$ integral hisoblansin.

$$\int_0^1 xe^{-x}dx = \left| \begin{matrix} u = x & du = dx \\ dv = e^{-x}dx & v = -e^{-x} \end{matrix} \right| = -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}dx = -e^{-1} - e^{-x}\Big|_0^1 =$$

$$= -e^{-1} - e^{-1} + 1 = 1 - \frac{2}{e};$$

Izoh: Ba'zi integrallarni hisoblashda bo'laklab integrallash formulasini bir necha marta qo'llash mumkin.

**3)** $\int_0^1 \arcsin xdx$ integral hisoblansin.[8]

## Aniq intagralning tatbiqlari.

### Figuralar yuzlarini dekart koordinatalar sistemasida hisoblash

a) Avvalgi o'tilgan mavzulardan ma'lumki, agar $[a, b]$ kesmada funksiya $f(x) \geq 0$ bo'lsa, u holda $y=f(x)$ egri chiziq, $OX$ o'qi va $x$=a hamda $x=b$ to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzi

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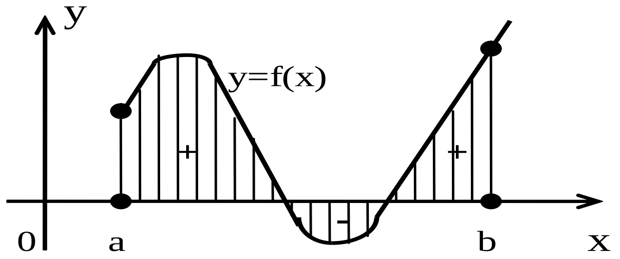
[8] *Larson Edvards. /Calculus/ 2010. 529,226 betlarning mazmun mohiyatidan foydalanildi.*

$$S=\int_a^b f(x)dx \qquad (4)$$

ga teng bo’ladi. Agar [*a*, *b*] kesmada *f*(*x*)≤0 bo’lsa, u holda aniq integral $\int_a^b f(x)dx \le 0$ bo’ladi.

Absolyut qiymatiga ko’ra bu integralning qiymati ham tegishli egri chiziqli trapetsiyaning yuziga teng:

$$S=\int_a^b |f(x)|dx \qquad (4^{I})$$




1-rasm

Agar *f*(*x*) funksiya [*a*, *b*] kesmada ishorasini chekli son marta o’zgartirsa, u holda integralni butun [*a*, *b*] kesmada qismiy kesmachalar bo’yicha integrallar yig’indisiga ajratamiz. *f*(*x*)>0 bo’lgan kesmalarda integral musbat, *f*(*x*)<0 bo’lgan kesmalarda integral manfiy bo’ladi. Butun kesma bo’yicha olingan integral *OX* o’qidan yuqorida va pastda yotuvchi yuzlarning tegishli algebraik yig’indisini beradi (1-rasm). Yuzlar yig’indisini odatdagi ma’noda hosil qilish uchun yuqorida ko’rsatilgan kesmalar bo’yicha olingan integrallar absolyut qiymatlari yig’indisini topish yoki

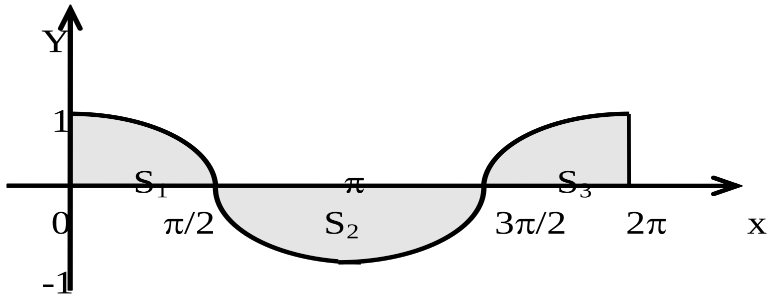
$$S=\int_a^b |f(x)|dx \qquad (4^{II})$$

integralni hisoblash kerak.

b) Agar $y_1=f_1(x)$ va $y_2=f_2(x)$ egri chiziqlar hamda *x*=*a* va *x*=*b* to’g’ri chiziqlar bilan chegaralangan figuraning yuzini hisoblash kerak bo’lsa, u holda $f_1(x)\ge f_2(x)$ shart bajarilgan figuraning yuzi qo’yidagiga teng:

$$S=\int_a^b (f_1(x)-f_2(x))dx \qquad (5)$$

***1-misol.*** *y*=cos*x*, *y*=0 chiziqlar bilan chegaralangan figuraning yuzi hisoblansin, bunda $x\in[0, 2\pi]$ (2-rasm).




2-rasm

***Yechish.*** $x\in[0, \pi/2]$ va $x\in[3\pi/2, 2\pi]$ da $\cos x\geq 0$ hamda $x\in[\pi/2, 3\pi/2]$ da $\cos x\leq 0$ bo'lgani uchun

$$S=\int_0^{2\pi}|\cos x|\,dx=\int_0^{\pi/2}\cos x dx+\left|\int_{\pi/2}^{3\pi/2}(-\cos x)dx\right|+\int_{3\pi/2}^{2\pi}\cos x dx=\sin x\Big|_0^{\pi/2}+$$

$$+\left|\sin x\Big|_{\pi/2}^{3\pi/2}\right|+\sin x\Big|_{3\pi/2}^{2\pi}=\sin\frac{\pi}{2}-\sin 0+\left|\sin\frac{3\pi}{2}-\sin\frac{\pi}{2}\right|+$$

$$+\sin 2\pi-\sin\frac{3\pi}{2}=1+|-1-1|-(-1)=4$$

Demak, $S$ =4(kv.birlik).

**Mavzu yuzasidan savol va topshiriqlar**

1. Aniq integralning aniqmas integraldan farqi nimada?
2. Aniq integralning ta'rifini ayting va tushuntiring.
3. Aniq integralning geometrik ma'nosini tushuntiring.
4. Aniq integralning xossalarini ayting.
5. N'yuton-Leybnits formulasini keltiring.
6. Aniq integralning tatbiqlarini misollarda tushuntiring.